Hysteresis and persistent currents in a rotating Bose-Einstein condensate with anharmonic confinement

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(February 13, 2004)

We examine a Bose-Einstein condensate of atoms that rotates in a quadratic-plus-quartic potential. It is shown that states of different circulation can be metastable. As a result, we demonstrate that the gas can exhibit hysteresis as the angular frequency of rotation of the trap is varied. The simplicity of the picture that emerges for small coupling strengths suggests that this system may be attractive for studies of phase transitions.

PACS numbers: 03.75.Hh, 03.75.Kk, 67.40.Vs

I. INTRODUCTION

The phenomenon of hysteresis is wide-spread in physics and can be expected whenever a physical system undergoes a first-order phase transition. Consider, for example, a system whose free energy has two minima as a function of its order parameter. As some external parameter is varied, these can shift with respect to each other. Provided only that fluctuations are not of a strength sufficient to bring the system to the absolute minimum of its free energy, it will remain at a local minimum even of this is not the absolute minimum, i.e., even if it is in a metastable state. The existence of metastable states implies that the actual state of the system depends on its history.

Superfluids number among the many physical systems which can display hysteresis. The rigidity against weak perturbations of the state of a superfluid that rotates in a bucket or flows in a pipe can be attributed to its metastability [1]. For this reason, these fluids are expected to show hysteresis and actually do so. For example, if such a fluid is inside a container which slowly starts to rotate, it remains stationary until the angular frequency reaches some critical value. In the reverse process, with the fluid in motion, the system follows a different path, and the fluid continues to rotate even when the container does not.

In the present paper we demonstrate how these ideas apply to a Bose-Einstein condensate of alkali-metal atoms that rotates in a quadratic-plus-quartic potential. Our motivation for this study comes from recent experiments in such trapping potentials which observed vortex states [2,3]. Using a simple but still realistic model [4,5], we demonstrate in Sec. II the existence of metastable states in these systems. In Sec. III we examine how the effects of hysteresis can appear as the frequency of rotation of the trap changes. In Sec. IV we discuss our results, and Sec. V summarizes the basic points of our study.

II. MODEL – STRUCTURE OF THE ENERGY SURFACE

To see how hysteresis works in this problem, it is useful to give a brief summary of the results of Refs. [4,5]. One of the most important elements of the present study is to understand the structure of the energy of the gas in the rotating frame of reference, $E' = E - \mathbf{L}\cdot\mathbf{\Omega}$, where E is the energy in the lab frame, $\mathbf{L}$ is its angular momentum, and $\mathbf{\Omega}$ is the frequency of rotation of the external trapping potential $V(\rho)$. This trapping potential is assumed to have the form

$$V(\rho) = \frac{1}{2}M\omega^2\rho^2[1+\lambda(\frac{\rho}{a_0})^2]. \quad (1)$$

Here, ρ is the cylindrical polar coordinate, M is the atomic mass, $a_0 = (\hbar/M\omega)^{1/2}$ is the oscillator length, and λ is a small dimensionless constant. (In the experiment of Ref. [2], $\lambda \approx 10^{-3}$.) The trapping along the axis of rotation, the z axis, has been neglected for simplicity, and the appropriate density is thus the number of atoms N per unit length Z, $\sigma = N/Z$.

The (repulsive) interaction between atoms is assumed to have the usual form

$$V_{\rm int} = \frac{1}{2}U_0 \sum_{i\neq j}\delta(\mathbf{r}_i - \mathbf{r}_j), \quad (2)$$

where $U_0 = 4\pi\hbar^2 a/M$ is the strength of the effective two-body interaction with a equal to the scattering length for atom-atom collisions. In the limit of weak interactions considered here, the typical atomic density n is $\sim N/(\pi a_0^2 Z)$. Thus, for a typical interaction energy nU_0, $nU_0/\hbar\omega \sim \sigma a$ and the dimensionless quantity which plays the role of a coupling constant in this two-dimensional problem is σa, which is assumed to be smaller than unity.

In the limit $\sigma a \ll 1$ and $\lambda \ll 1$ the appropriate basis states with angular momentum $m\hbar$ are the eigenstates of the harmonic potential with no radial nodes,

$$\Phi_m(\rho,\phi) = \frac{1}{(m!\pi a_0^2 Z)^{1/2}}\left(\frac{\rho}{a_0}\right)^{|m|} e^{im\phi}e^{-\rho^2/2a_0^2}, \quad (3)$$

where ϕ is the angle in cylindrical polar coordinates. Quite generally, the order parameter, Ψ, of the condensate can then be expanded in this basis as

$$\Psi = \sum_m c_m \Phi_m. \tag{4}$$

It was shown in Ref. [4] that, for sufficiently small σa, the energy of the gas in the rotating frame, E', is minimized when only one component $m = m_0$ in the expansion of Eq. (4) contributes to the order parameter, $\Psi = \Phi_{m_0}$. In addition, the critical frequencies which denote the lower limit on the *absolute* stability of the states Φ_{m_0} are given by

$$\frac{\Omega_{m_0}}{\omega} = \frac{m_0}{|m_0|}\left[1 + \lambda(|m_0|+1) - \sigma a \frac{(2|m_0|-2)!}{2^{2|m_0|-1}(|m_0|-1)!|m_0|!}\right]. \tag{5}$$

The solid (straight) line in Fig. 1 shows this boundary between the states with $m_0 = 0$ and $m_0 = 1$. On the left of this line the energy E' has an absolute minimum for $\Psi = \Phi_0$; on the right the absolute minimum occurs for $\Psi = \Phi_1$. This is shown schematically in Fig. 2, where the minimum on the left in the graphs corresponds to the state Φ_0, while the one on the right corresponds to the state Φ_1.

As discussed in Refs. [4,5], there are metastable states for certain values of σa and Ω/ω in addition to these absolute minima. Consider, for example, the case $m_0 = 0$ of a non-rotating condensate. (The same procedure can be applied to any other m_0.) While the state Φ_0 represents the absolute minimum of E' in regions I and II, it is metastable in region III and unstable in region IV with respect to the state

$$\Psi = c_{-1}\Phi_{-1} + c_0\Phi_0 + c_1\Phi_1 \tag{6}$$

for $\sigma a < 0.06$. For $\sigma a > 0.06$ (but σa still sufficiently small) the instability is towards the state

$$\Psi = c_{-2}\Phi_{-2} + c_0\Phi_0 + c_2\Phi_2. \tag{7}$$

These genuine instabilities are given by the right dashed line in Fig. 1.

To be specific, the energy E' of the system per particle in the rotating frame in the state of Eq. (6) is

$$\begin{aligned}\frac{E'}{N\hbar\omega} &= |c_{-1}|^2(1+\Omega/\omega) + |c_1|^2(1-\Omega/\omega) \\ &\quad + \lambda(3|c_{-1}|^2 + |c_0|^2 + 3|c_1|^2) \\ &\quad + \sigma a(\frac{1}{2}|c_{-1}|^4 + |c_0|^4 + \frac{1}{2}|c_1|^4 + 2|c_{-1}|^2|c_0|^2 \\ &\quad + 2|c_0|^2|c_1|^2 + 2|c_{-1}|^2|c_1|^2 - 2|c_{-1}||c_0|^2|c_1|).\end{aligned} \tag{8}$$

Making use of the normalization condition, E' can be expressed in terms of c_0 and c_1, and is a two-dimensional energy surface. As one crosses the line $\Omega/\omega = 1 + 2\lambda - \sigma a/2$ (i.e., the straight solid line in Fig. 1 given by Eq. (5) for $m_0 = 0$), the absolute minimum moves discontinuously from $|c_0|^2 = 1$ to $|c_1|^2 = 1$. Furthermore, even above this line, there is a region in the $\Omega/\omega - \sigma a$ plane (i.e., region III in Fig. 1), for which E' has a local minimum at $|c_0|^2 = 1$. The state Φ_0 is thus an absolute minimum of E' in regions I and II, and it describes a local minimum in E' in region III, i.e., it is metastable. In region IV, on the other hand, the local minimum disappears. These observations are shown schematically in Fig. 2, where the minimum on the left corresponds to Φ_0.

Similarly, for $\Psi = \Phi_1$, the left dashed line in Fig. 1 gives the boundary for the (local) stability of the state Φ_1 with respect to the state

$$\Psi = c_0\Phi_0 + c_1\Phi_1 + c_2\Phi_2. \tag{9}$$

In regions III and IV, Φ_1 represents the absolute minimum in E', in region II it is metastable, and in region I it no longer represents a local minimum. These facts are also shown schematically in Fig. 2, where the minimum on the right corresponds to Φ_1.

The general picture of stability/metastability that emerges for all $m_0 \geq 2$ is shown as the dashed line in Fig. 3 for $m_0 = 2$. The state Φ_2 provides the absolute minimum of E' between the straight lines, while in the remaining part, Φ_2 is metastable. Along the dashed curve, and as one moves clockwise, the instability is against a linear combination of states involving the following m: $(1,2,3), (0,2,4), (-2,2,6), (-1,2,5), (0,2,4)$, and $(1,2,3)$.

III. HYSTERESIS

With these observations, it is easy to demonstrate the effect of hysteresis. Assume that the condensate is initially at rest and that Ω increases slowly from zero with σa kept fixed (and less than 0.06 for simplicity.) So long as Ω/ω is in the regions I and II, the cloud will not rotate since E' has its absolute minimum in the state with $m_0 = 0$. Provided that fluctuations are sufficiently weak, the system will not rotate even in region III since Φ_0 is a metastable state. The gas will begin to rotate only when Ω/ω is in region IV, where Φ_0 is unstable against the state of Eq. (6). The system will then move to the pure state Φ_1 (i.e., the absolute minimum of the system) on the fastest available time scale. In this process, it is important to note that the gas is unstable with respect to the state of Eq. (6) in region IV. Since Φ_1 is one of the components of Ψ, there is a monotonically-decreasing path in the energy from the (unstable) pure state Φ_0 to the pure state Φ_1, which is the absolute minimum of E'.

If one now considers the inverse process in which Ω decreases slowly from region IV with the gas initially in the state Φ_1, the system will remain in this state in regions IV and III where it is the absolute minimum and in region II where Φ_1 is metastable. In the absence of

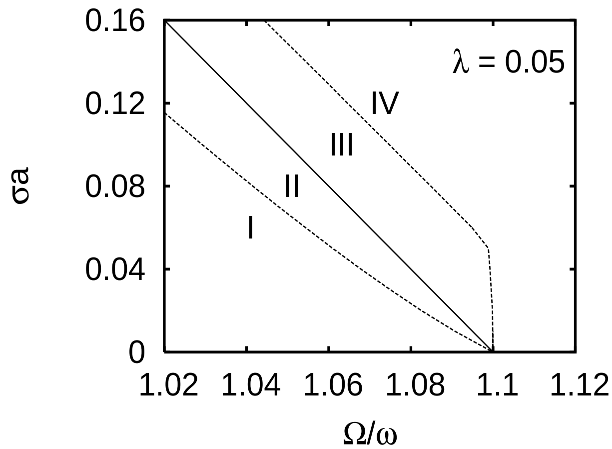

FIG. 1. The phase diagram of a rotating Bose-Einstein condensate confined in a quadratic-plus-quartic potential in the $\Omega/\omega - \sigma a$ plane. The straight solid line gives the phase boundary between the phases where the states Φ_0 (regions I, II) and Φ_1 (regions III, IV) provide the absolute minima of the energy in the rotating frame. Φ_0 is metastable in region III and unstable in region IV. Similarly Φ_1 is metastable in region II and unstable in region I.

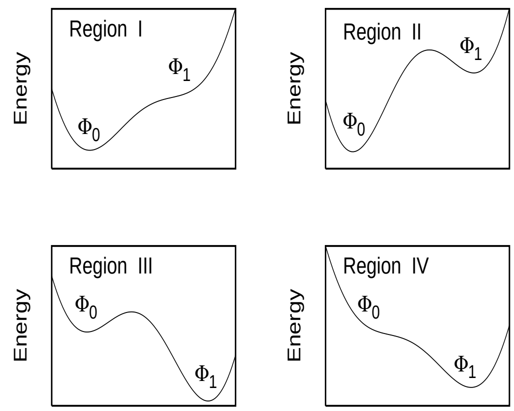

FIG. 2. Schematic diagram of the energy of the system in the rotating frame as a function of the order parameter for a fixed σa and for values of Ω/ω in regions corresponding to those in Fig. 1. The minimum on the left corresponds to the state Φ_0; that on the right to the state Φ_1.

significant fluctuations, a transition will occur only when Ω/ω reaches region I. In this region, Φ_1 is unstable to the state of Eq. (9), which includes a component of Φ_0. The system will move to the absolute minimum described by the (pure) state Φ_0, and rotation will cease. The gas thus displays hysteresis in regions II and III. The order parameter is given by Φ_0 as Ω increases; by Φ_1 as it decreases.

A similar process evidently exists for larger values of Ω, and we show the first few steps of the angular momentum m of the gas as a function of Ω/ω in Fig. 4 for $\lambda = 0.05$ and $\sigma a = 0.05$. The picture that emerges for small values of σa is clear, since the leading instability of a given state Φ_{m_0} (for increasing/decreasing Ω) always involves a linear superposition of the form

$$\Psi = c_{m_0-n}\Phi_{m_0-n} + c_{m_0}\Phi_{m_0} + c_{m_0+n}\Phi_{m_0+n}, \qquad (10)$$

with $n = 1$. (For $\sigma a = 0$ the equation that yields each solid curve coincides with those leading to the corresponding dashed curves [4,5]. Thus, the two dashed curves always meet their solid partner at $\sigma a = 0$.) As soon as Φ_{m_0} becomes unstable, the order parameter will adjust rapidly to the new minimum provided by the state $\Phi_{m_0 \pm 1}$.

We have found that the largest value of σa, $(\sigma a)_c$, for which the instability of a given Φ_{m_0} is to a state of the form of Eq. (10) with $n = 1$ increases roughly linearly with m [$(\sigma a)_c = 0.06$ for $m_0 = 0$]. In this case E' can have at most two local minima. The transition between different states as Ω increases/decreases is then clear and consists of steps in the angular momentum m of the gas equal to ± 1. For fixed σa, the pattern shown in Fig. 4 will always hold if Ω/ω is sufficiently large since $(\sigma a)_c$ increases with m and σa will eventually become smaller than $(\sigma a)_c$.

For values of $\sigma a > (\sigma a)_c$ the situation becomes more rich, since there are instabilities to many different combinations of states, depending on the specific value of σa [like, for example, the one in Eq. (7)] of the general form given in Eq. (10), with n being a integer. In this case there is a wide variety of possibilities where the system will evolve as Ω changes. The (multi-dimensional) energy E' develops many local minima and small changes in σa may drive the gas to completely different states, as it will follow the path with the steepest decrease in its energy. The corresponding graph of Fig. 4 is then in principle more complicated, as the steps in $|m|$ may be different than unity and can be non-integer. When the absolute minimum energy is given by Eq. (10) with all coefficients non-zero, the order parameter has an n-fold *discrete* rotational symmetry [5]. This observation is of some importance. The existence of metastable states is a necessary condition for hysteresis. As repeatedly noted, the ability to observe hysteresis also requires the absence of fluctuations sufficient to cause phase transitions from metastable states. The fluctuations required to change discrete rotational symmetry become relatively improbable and the experimental indications of hysteresis are likely to be clearer for larger values of σa where one or both states are given by Eq. (10) [5].

IV. DISCUSSION

Reference [6] has shown that similar effects of hysteresis also appear in harmonically-trapped Bose-Einstein

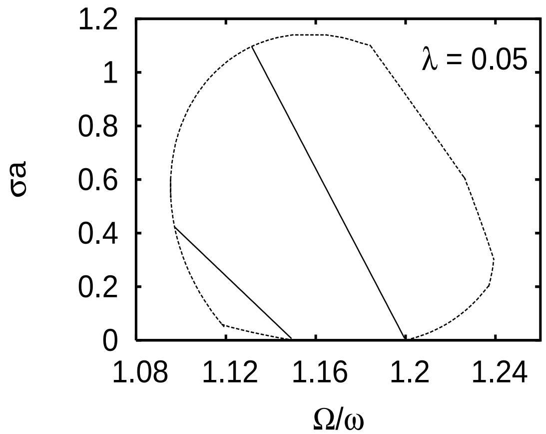

FIG. 3. Regions of absolute stability/metastability of the state Φ_2. Between the solid lines Φ_2 provides the absolute minimum of E', while in the remaining region enclosed by the dashed curve it is metastable.

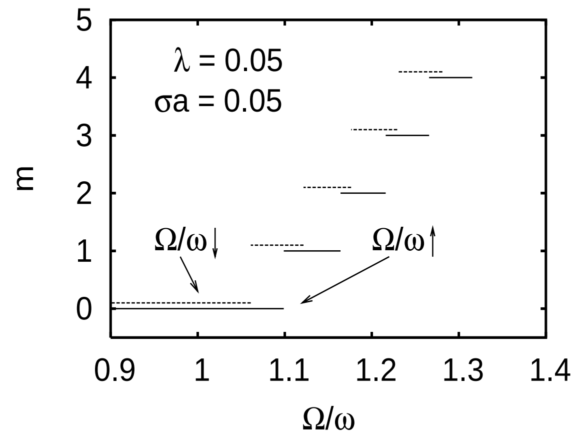

FIG. 4. Demonstration of hysteresis. The graph shows the angular momentum, m, of the system as function of the frequency of the rotation of the trap, Ω/ω, as Ω/ω increases (solid lines) and as it decreases (dashed lines). The dashed lines have been displaced slightly. Here, $\lambda = 0.05$ and $\sigma a = 0.05$.

condensates. (See also [7].) This is no surprise since hysteresis is a general feature of superfluids. These results are also consistent with the corresponding experimental observations of Ref. [8].

One important aspect of our study, which also applies in the case of harmonic trapping, is that the critical frequencies for the creation of vortex states observed in such experiments depend on how the experiment is performed. For example, if one first cools the gas in the condensed phase and then rotates the trap, the picture that emerges is that of Fig. 4. If, on the other hand, the trap is first set in motion and the gas then cooled, it will end up at the absolute minima of E' as given by Eq. (5). (See Fig. 1 in Ref. [4].)

The simple structure of E' for small values of σa described here is an interesting feature, and this system could prove to be an ideal laboratory for the investigation of the effects of quantum and thermal fluctuations on phase transitions. It is important to mention that the boundaries we have calculated here are exact in the limit of small σa. As in other problems that involve phase transitions between metastable states, it is assumed that fluctuations are suppressed because of the energy barrier that separates the two states. In the presence of such an energy barrier, the characteristic time scale for the phase transition is expected to be long — particularly when the two local minima have distinct discrete symmetry. When the barrier disappears and the metastable state becomes unstable, the phase transition proceeds rapidly. Here, we have suggested an essentially exact description of the phase boundaries and the corrersponding energy surface.

The effects considered here could be useful for (i) probing persistent currents in superfluids, (ii) providing experimental information regarding the energy surface of the system, and (iii) investigating the effect of fluctuations on the phase transitions (e.g., by varying either the number of atoms or the temperature).

V. SUMMARY

In summary, we have investigated the energy of a Bose-Einstein condensate that rotates in a quadratic-plus-quartic trapping potential. The energy surface is elementary in the limit of small coupling with a pattern of absolutely stable and metastable states which can give rise to potentially observable persistent currents and hysteresis phenomena. Finally, because of its simplicity, the present system can offer some insight into the physics of phase transitions that are driven by fluctuations.

[1] A. J. Leggett, Rev. of Mod. Phys., **73**, 307 (2001).

[2] V. Bretin, S. Stock, Y. Seurin, and J. Dalibard, e-print cond-mat/0307464.

[3] S. Stock, V. Bretin, F. Chevy, and J. Dalibard, e-print cond-mat/0311099.

[4] A. D. Jackson and G. M. Kavoulakis, e-print cond-mat/0311066.

[5] A. D. Jackson, G. M. Kavoulakis, and E. Lundh, e-print cond-mat/0401292.

[6] J. J. García-Ripoll and V. M. Pérez-García, Phys. Rev. A **63**, 041603 (2001).

[7] E. J. Mueller, Phys. Rev. A **66**, 063603 (2002).

[8] K. W. Madison, F. Chevy, W. Wohlleben, and J. Dalibard, Phys. Rev. Lett. **84**, 806 (2000); F. Chevy, K.W. Madison, and J. Dalibard, Phys. Rev. Lett. **85**, 2223 (2000); K. W. Madison, F. Chevy, V. Bretin, and J. Dalibard, Phys. Rev. Lett. **86**, 4443 (2001).